Renormalization of earthquake aftershocks

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Abstract

Assume that each earthquake can produce a series of aftershock independently of its size according to its "local" Omori's law with exponent $1+\theta$. Each aftershock can itself trigger other aftershocks and so on. The global observable Omori's law is found to have two distinct power law regimes, the first one with exponent $p_- = 1-\theta$ for time $t < t^* \sim \kappa^{-1/\theta}$, where $0 < 1-\kappa < 1$ measures the fraction of triggered earthquakes per triggering earthquake, and the second one with exponent $p_+ = 1+\theta$ for larger times. The existence of these two regimes rationalizes the observation of Kisslinger and Jones [1991] that the exponent p seems positively correlated to the surface heat flow: a higher heat flow is a signature of a higher crustal temperature, which leads to larger strain relaxation by creep, corresponding to fewer events triggered per earthquake, i.e. to a larger κ, and thus to a smaller t^*, leading to an effective measured exponent more heavily weighted toward $p_+ > 1$.

Index terms: 3200 MATHEMATICAL GEOPHYSICS, 3220 Nonlinear dynamics, 7209 Earthquake dynamics and mechanics, 7260 Theory and modeling

I. INTRODUCTION

Aftershocks constitute a very significant percentage of even the large shallow earthquakes [*Hough and Jones*, 1997] and all the more so for the small earthquakes that dominate the catalogs. An aftershock is usually defined as any earthquake that occurs within a distance equal to the length of the fault that ruptured during the mainshock and during the span of time that seismicity rate in that region remains above its pre-mainshock background level.

This definition can in fact be imprecise for a variety of reasons :

1. the pre-mainshock rate can be ill-defined due to inherent intermittent behavior [*Kagan and Jackson*, 1991; *Kagan*, 1994] or poorly constrained by sparse data. The intermittency or clustering in time is sometimes quantified by fractal tools [*Dattatrayam and Kamble*, 1994; *Henderson*, 1994].

2. Some earthquakes are sometimes triggered hundreds of kilometers away, at significantly greater distance than the mainshock rupture length [*Hill et al.*, 1993; *Steeples and Steeples*, 1996].

3. Patterns of spatial and temporal self-organization between events at large distance and time scales have been observed [*Knopoff et al.*, 1997; *Bowman et al.*, 1998] that suggest more complex interactions than a simple time delay mechanism between mainshock and aftershocks, and has been ascribed to a kind of self-organized critical behavior [*Sornette and Sornette*, 1989; *Bak and Tang*, 1989; *Grasso and Sornette*, 1998];

4. In some models of self-organized spatio-temporal earthquake behavior [*Huang et al.*, 1998; *Hainz et al.*, 1998], the same physical mechanism is found to produce both the Gutenberg-Richter frequency-magnitude distribution and the foreshock and aftershock Omori's laws.

The identification of an aftershock becomes feasible, by definition, only after an event has been recognized as a main shock. This main shock can itself be an aftershock of a larger event that preceded it. The time-delayed interactions between mainshock and aftershocks, that are usually invoked [*Scholz*, 1990; *Das and Scholz*, 1981; *Shaw*, 1993], have no reason not to be at work between two events when the second one is the large. Kagan and Knopoff [1978] have even asserted that both foreshocks and aftershocks are a manifestation of essentially the same process.

These considerations lead us to formulate the following hypothesis and explore its self-consistency :

Hypothesis : extending the usual definition of an aftershock to allow for the fact that the delayed triggered events can be of arbitrary size and not just smaller, can we construct a coherent model of aftershocks in which each earthquake is the generator of a sequence of delayed triggered events of arbitrary size, in other words, each earthquake is the "generalized" aftershock of a some preceding event?

To address this question, we explore the following model. Each event triggers its own "local" Omori's rate of aftershocks. Each of these aftershocks itself triggers its own "local" Omori's law of aftershock, and so on in an infinite branching process. The relevant question for observations is what is the resulting global Omori's law? Indeed, the individual "local" Omori's rate triggered by each event is not observable, only the superposition of all events at all levels of the cascade is usually quantified. This problem belongs to the general class in which one specifies the laws at the "microscopic" scale, here the local Omori's law, and strives to derive the resulting "macroscopic" laws. In this paper, we do not address the mechanism(s) at the origin of the "local" Omori's law but just take it for granted. We rather address the collective behavior, i.e. how the "local" law is "renormalized" [*Wilson*, 1979] into a macroscopic observable Omori's law.

II. SELF-CONSISTENT OMORI'S LAW

To formulate the question in formal terms, we follow Kagan and Knopoff (1981) and assume that the probability that one event occurring at time zero gives birth to another in the time interval between t and $t + dt$ is

$$\phi(t)dt = (1 - \kappa)\ \theta\ t_0^\theta\ \frac{1}{t^{1+\theta}}\ H(t - t_0), \qquad (1)$$

where $H(t)$ is the Heaviside function: $H(t - t_0) = 0$ for $t < t_0$ and 1 otherwise, which means that the triggering of seismic activity starts a short delay t_0 after the shock. This minimum waiting time is necessary for the total number of events to be finite, otherwise $\phi(t)$ would not be normalized due to the divergence at $t \to 0$. Physically, it represents the simplest way to account for a progressive build up of activity after the main shock.

This expression is exactly Omori's law for the rate of aftershocks following a main shock, albeit with the modification that we do not specify that aftershocks have to be smaller. The exponent $1 + \theta$ of the "local" Omori's law has not reason a priori to be the same as the one measured macroscopically which is usually found between 0.8 and 1.2 with an often quoted value 1. This is in fact the question we address : assuming the form (1) for the "local" Omori's law, is the global Omori's law still a power law and, if yes, how does its exponent depend on θ?

The integral of $\phi(t)$ over time is the average number of earthquakes created by each event and is equal to

$$\int_0^{+\infty} \phi(t)\ dt = 1 - \kappa. \qquad (2)$$

If $\kappa < 0$, more than one earthquake is triggered per earthquake. This regime corresponds to the super-critical regime of branching processes [*Harris*, 1963] in which the total number of events grows exponentially with time. If $\kappa > 0$, there is less than one earthquake triggered per earthquake. This is the sub-critical regime in which the number of events following the first main shock decays eventually to zero. The critical case $\kappa = 0$ is at the borderline between the two regimes. In this case, there is exactly one earthquake on average triggered per earthquake and the process is exactly at the critical point between death on the long run and exponential proliferation. In the sequel, we take $\kappa > 0$ as the physically relevant regime where the aftershock activity following a main shock eventually decays at long times.

We analyze the case where there is an origin of time $t = 0$ at which we start recording the rate of earthquakes, assuming that the largest earthquake of all has just occurred at $t = 0$ and somehow reset the clock. In the following calculation, we will forget about the effect of events preceding the one at $t = 0$ and count aftershocks that are created only by this main shock.

Let us call $N(t)dt$ the number of earthquakes between t and $t+dt$. $N(t)$ is the solution of a self-consistency equation that formalizes mathematically the following process : the first earthquake may trigger aftershocks; these aftershocks may trigger their own aftershocks, and so on. The rate of seismicity at a given time t is the result of this cascade process. The self-consistency equation that sums up this cascade reads

$$N(t) = \int_0^t N(\tau)\phi(t-\tau)d\tau \ . \tag{3}$$

Its meaning is the following. The rate $N(t)$ at time t is the sum over all induced rates from all earthquakes that occurred at all previous times with an induced rate per earthquake occurring at time $t-\tau$ equal to $\phi(t-\tau)$. The essential point is that the *same* function N appears on the l.h.s. and in the integrant in the r.h.s.

The lower bound 0 in the integral in (3) expresses the fact that we look at the time development of the seismic activity after an origin of time, that can be taken for instance as the triggering time at which a big event occurred. Such an origin of time is important as we are not describing a stationary process, but rather a rate which on average relaxes (slowly) to zero.

The problem is then to determine the functional form of $N(t)$, assuming that ϕ is given by (1). The integral equation (3) is a Wiener-Hopf integral equation [*Feller*, 1971]. It is well-known [*Feller*, 1971; *Morse and Feshback*, 1953] that, if $\phi(\tau)$ decays no slower than an exponential, then $N(t)$ has an exponential tail $N(t) \sim \exp[-\mu t]$ for large t with μ solution of $\int \phi(x)\ \exp[\mu x]\ dx = 1$. In the present case, $\phi(\tau)$ decays much slower than an exponential and a different analysis is called for that we now present.

III. SOLUTION OF THE WIENER-HOPF INTEGRAL EQUATION FOR POWER LAW KERNEL

Inserting (1) in (3) gives

$$N(t) = (1-\kappa)\ \int_0^{t-t_0} N(\tau)\ \theta\ t_0^\theta\ (t-\tau)^{-(1+\theta)}\ d\tau. \tag{4}$$

The upper bound $t - t_0$ comes from the Heaviside function and reflects the rule (1) that the rate $N(t)$ at t is influenced only by events that occurred at least a time t_0 before.

By writing (4) for time $t + t_0$, we get

$$N(t+t_0) = (1-\kappa)\int_0^t N(\tau)\ K_{t_0}(t-\tau) d\tau\ , \tag{5}$$

where we define the Kernel

$$K_{t_0}(x) \equiv \theta\ t_0^\theta\ (x+t_0)^{-(1+\theta)}\ . \tag{6}$$

We recognize that the r.h.s. of (5) is exactly the convolution of $N(t)$ with $K_{t_0}(t)$. Since we are dealing with a non-stationary process (i.e. there is an origin of time) and we have a convolution operator, the natural tool is the Laplace transform $\hat{f}(\beta) \equiv \int_0^{+\infty} f(t) e^{-\beta t} dt$. Applying the Laplace transform to (6) yields

$$\int_0^{+\infty} N(t+t_0) e^{-\beta t} dt = (1-\kappa)\hat{N}(\beta)\hat{K}_{t_0}(\beta)\ , \tag{7}$$

where the r.h.s. has used the convolution theorem. To deal with the l.h.s., we transform it as

$$\int_0^{+\infty} N(t+t_0)e^{-\beta t}dt = e^{\beta t_0}\int_{t_0}^{+\infty} N(t)e^{-\beta t}dt = e^{\beta t_0}\Big(\int_0^{+\infty} N(t)e^{-\beta t}dt - \int_0^{t_0} N(t)e^{-\beta t}dt\Big). \tag{8}$$

Recall that the first great event is assumed to occur at $t = 0$. During the time interval from 0 to t_0, no other event occurs. Thus, $\int_0^{t_0} N(t)e^{-\beta t}dt = t_0 N(0)$ with $N(0) = 1$. We also make the approximation $e^{\beta t_0} \approx 1$, based on the condition that we look at earthquake rates at long times $t_0 << t$ which translates into the condition $\beta << t_0^{-1}$ for the Laplace conjugate variable β. We thus get

$$\int_0^{+\infty} N(t+t_0)e^{-\beta t}dt \approx \hat{N}(\beta) - t_0 N(0)\ . \tag{9}$$

Reporting into (7) and solving for $\hat{N}(\beta)$ gives the expression of the Laplace transform of $N(t)$:

$$\hat{N}(\beta) = \frac{t_0 N(0)}{1-(1-\kappa)\hat{K}_{t_O}(\beta)}. \tag{10}$$

The solution for $N(t)$ is derived by taking the inverse Laplace transform of (10).

In this goal, we shall now make use of a well-known result for the expression of the Laplace transform of a power law. The important point to notice is that $K(t) \sim t^{-(1+\theta)}$ for large t. Then, its Laplace transform takes the form (obtained by integrating by part l times, where l the integer part of θ *i.e.* $l < \theta < l+1$)

$$\hat{K}(\beta) = e^{-\beta}\Big(1 - \frac{\beta}{\theta-1} + ... + \frac{(-1)^l \beta^l}{(\theta-1)(\theta-2)...(\theta-l)}\Big) +$$
$$+ \frac{(-1)^l \beta^\theta}{(\theta-1)(\theta-2)...(\theta-l)} \int_\beta^\infty dx e^{-x} x^{l-\theta}. \tag{11}$$

This last integral is equal to

$$\beta^\theta \int_\beta^\infty dx e^{-x} x^{l-\theta} = \Gamma(l+1-\theta)[\beta^\theta + \beta^{l+1}\gamma^*(l+1-\theta,\beta)], \tag{12}$$

where Γ is the Gamma function ($\Gamma(n+1) = n!$) and

$$\gamma^*(l+1-\theta,\beta) = e^{-\beta}\sum_{n=0}^{+\infty}\frac{\beta^n}{\Gamma(l+2-\theta+n)} \tag{13}$$

is the incomplete Gamma function. We see that $\hat{K}(\beta)$ presents a regular Taylor expansion in powers of β up to the order l, followed by a term of the form β^θ. We can thus write

$$\hat{K}(\beta) = 1 + r_1\beta + ... + r_l\beta^l + r_\theta\beta^\theta + \mathcal{O}(\beta^{l+1}), \tag{14}$$

with $r_1 = -\langle t\rangle,\ r_2 = \frac{\langle t^2\rangle}{2}, ...$ are the moments of K. For small β, we rewrite $\hat{K}(\beta)$ under the form

$$\hat{K}(\beta) = \exp\left[\sum_{k=1}^{l} d_k\beta^k + d_\theta\beta^\theta\right], \tag{15}$$

where the coefficient d_k can be simply expressed in terms of the r_k's. We recognize that this transformation is similar to that from the moments to the cumulants. The expression (15) generalizes the canonical form of the characteristic function of the stable Lévy laws [*Gnedenko and Kolmogorov*, 1954] for arbitrary values of θ, and not solely for $\theta \leq 2$ for which they are defined. The canonical form is recovered for $\theta \leq 2$ for which the coefficient d_2 is not defined (the variance does not exist) and the only analytical term is $\langle t\rangle\beta$ (for $\mu > 1$).

Here, we are interested in the case $0 < \theta < 1$ for which

$$\hat{K}_{t_O}(\beta) = 1 - d(\beta t_0)^\theta + h.o.t. \tag{16}$$

where *h.o.t.* stands for higher order terms. Reporting in (10) this gives

$$\hat{N}(\beta) = \frac{t_0 N(0)}{1-(1-\kappa)[1-d(\beta t_0)^\theta]} = \frac{t_0 N(0)}{\kappa + d(1-\kappa)(\beta t_0)^\theta}. \tag{17}$$

Two cases must be distinguished.

- $(\beta t_0)^\theta << \kappa$ corresponds to $t >> t^* \equiv \frac{t_0}{\kappa^{\frac{1}{\theta}}}$. In this case, we can expand $\frac{t_0 N(0)}{\kappa + d(1-\kappa)(\beta t_0)^\theta}$, which leads to

$$\hat{N}(\beta) = \frac{t_0 N(0)}{\kappa}[1 - d\frac{1-\kappa}{\kappa}(\beta t_0)^\theta]. \tag{18}$$

We recognize the Laplace transform (15) of a power law of exponent θ, *i.e.*

$$N(t) \sim t^{-(1+\theta)} \qquad \text{for } t >> t^* \ . \tag{19}$$

- For $t < t^*$, $d(1-\kappa)(\beta t_0)^\theta > \kappa$ and (17) can be written with a good approximation as

$$\hat{N}(\beta) = \frac{t_0 N(0)}{d(1-\kappa)(\beta t_0)^\theta} \sim (\beta t_0)^{-\theta}. \tag{20}$$

Denoting $\Gamma(z) \equiv \int_0^{+\infty} dt \ e^{-t} \ t^{z-1}$, we see that $\int_0^{+\infty} dt \ e^{-\beta t} \ t^{z-1} = \Gamma(z)\beta^{-z}$. Comparing with (20), we thus get

$$N(t) \sim t^{-(1-\theta)} \qquad \text{for } t < t^* \ . \tag{21}$$

IV. DISCUSSION

Treating all aftershocks on the same footing by assuming that each may trigger new aftershocks with the same "local" rate gives a global Omori's law $N(t)$ described by *two* power laws with exponents $p_- = 1-\theta$ for $t < t^*$ and $p_+ = 1+\theta$ for $t > t^*$, where the characteristic time $t^* = t_0/\kappa^{1/\theta}$.

- Our results shows that the simple and parsimonious model in which all earthquakes are put on the same footing, i.e. are all susceptible to trigger its train of aftershocks is fully consistent with empirical observations. It suggests that the usual taxonomy identifying aftershocks as special events may not resist a more physically-based analysis.

- The value $\theta \rightarrow 0$, corresponding to a local Omori's exponent equal to one, is the only value that gives a completely self-similar rate, i.e. the same power law for all times, and in addition the same decay rate at the "local" and global level. In other words, the $1/t$ law is the only one that solves the self-consistent condition that the same law describes all levels of aftershock triggering processes.

• In practice, even a large deviation from $\theta = 0$, say, $\theta = 1/2$ may produce a reasonable approximation to the usual $1/t$ Omori's power law, due to the existence of the long cross-over between the two power laws whose exponents possess the value 1 as their exact barycenter. In this picture, the observed Omori's law is not exactly $1/t$ but a mixture of two power laws slowly crossing over from one to the other. This can also provide a mechanism for the variations of measured Omori's exponent [*Kisslinger and Jones*, 1991].

• Let us take $\theta = 0.2$ and $\kappa = 10^{-1}$. This yields $t^* = 10^5\ t_0$ for the characteristic time scale. If $t_0 = 1\ s$, then $t^* \approx 1$ day. Suppose now that $\kappa = 0.05$. This yields $t^* = 3.2\ 10^6\ t_0 \approx 30$ days. Thus, a small variation of the fraction $1 - \kappa$ of earthquakes triggered per earthquake (from 0.9 to 0.95) leads to a dramatic variation (day to month) of the characteristic time scale t^*. This offers a simple and physically appealing explanation for the observations of Kisslinger and Jones [1991] that Omori's exponent p seems to be positively correlated to the surface heat flow: since a higher heat flow is probably a signature of a higher crustal temperature, this leads to larger strain relaxation by creep. As a consequence, less strain is accomodated by earthquakes and there are thus fewer events triggered per earthquake, meaning that κ is larger and t^* is smaller. Now, the effective exponent p measured over a time interval t is approximately an average of p_- and p_+ weighted respectively by $t^* - t_0$ and $t - t^*$. If t^* decreases (resp. increases), the effective exponent p increases toward p_+ (resp. decreases toward p_-). Our theory thus views Omori's exponent as a direct gauge of the fraction of strain accomodated by earthquakes. This prediction can be tested by future investigations.

• Kagan and Knopoff [1981] have used a similar model to the one studied in the present paper in order to generate synthetic catalogs. Their model differs in the definition of the events, in the introduction of an additional exponential relaxation and in the introduction of a threshold for detection. These complications do not allow for an analytic treatment as performed here but their numerical simulations are compatible with our results, in particular with respect to the appearance of a long time scale $t^* = t_0/\kappa^{1/\theta}$.

We acknowledge stimulating discussions with Y. Kagan and L. Knopoff.

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